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A Note on Certain Divisibility Problem

C. Adiga and H. N. Ramaswamy

Department of Studies in Mathematics, University of Mysore
Manasagangothri, Mysore - 570 006, India
c_adiga@hotmail.com
hnrama@gmail.com

Abstract

In this note we investigate the positive integers n for which $\phi(n^2)+\sigma_2(n)$ is divisible by n^2.

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1 Introduction

Let $\phi(n)$ and $\sigma(n)$ denote the Euler function and sum of divisors function of n. More generally $\sigma_k(n)$ denotes the sum of k-th powers of positive divisors of n. Clearly n is prime if and only if n divides $\phi(n)+\sigma(n)$ and in fact $2n=\phi(n)+\sigma(n)$. C.A. Nicol [1] has studied the positive integers n which divide $\phi(n)+\sigma(n)$. Let $\mathbb{A}$ be given by

$$\mathbb{A}=\{n \text{ composite} : n|\,(\phi(n)+\sigma(n))\}.$$

Nicol proved that no square free integer belongs to $\mathbb{A}$ and conjectured that $\mathbb{A}$ contains only odd integers. M. Zhang [2] proved that $\mathbb{A}$ contains no integer of the form $p^\alpha q$, where p and q are distinct primes and α is a positive integer. Although F. Luca and J. Sandor [3] could not solve this conjecture, they have made a significant progress. In [3] they showed, among many other related results, that, for any fixed positive integer $k\geq 2$ there are only finitely many odd positive integers $n\in\mathbb{A}$ with $\omega(n)=k$ where $\omega(n)$ denotes the number of distinct prime divisors of n.

In this note we study a variant of the above problem: we look at those integers $n>1$ for which n^2 divides $\phi(n^2)+\sigma_2(n)$. Let $\mathbb{B}$ be given by

$$\mathbb{B}=\{n>1 : n^2|\,(\phi(n^2)+\sigma_2(n))\}.$$

We prove that if $n \in \mathbb{B}$, then $\omega(n) \geq 4$.

2 Main Results

In this section, we establish that there is no integer $n > 1$ with $\omega(n) = 1, 2$ or 3 such that $n^2|(\phi(n^2) + \sigma_2(n))$ thus proving that if $n \in \mathbb{B}$, then $\omega(n) \geq 4$. Note that $\phi(n^2) + \sigma_2(n) > n^2$ for all n. We prove that $\phi(n^2) + \sigma_2(n) < 2n^2$ for all n such that $\omega(n) \leq 3$. We recall that

$$\varphi(n) = n \prod_{p|n}(1 - 1/p) \text{ and } \sigma_2(n) = \prod_{p^r \| n} \frac{p^{2r+2} - 1}{p^2 - 1}.$$

Theorem 2.1. Let $n > 1$ with $\omega(n) = 1$. Then, $n^2 \nmid (\phi(n^2) + \sigma_2(n))$.

Proof. Suppose $n = p^r$. Then,

$$\begin{aligned}
\frac{\phi(n^2) + \sigma_2(n)}{n^2} &= \frac{\phi(n)}{n} + \frac{\sigma_2(n)}{n^2} \\
&= (1 - \frac{1}{p}) + \frac{p^{2r+2} - 1}{p^2 - 1} \times \frac{1}{p^{2r}} \\
&< 1 - \frac{1}{p} + \frac{p^2}{p^2 - 1} \\
&= 2 + \left(\frac{1}{p^2 - 1} - \frac{1}{p}\right) \\
&< 2, \quad \text{since} \quad \frac{1}{p} > \frac{1}{p^2 - 1}.
\end{aligned}$$

Thus, $n^2 \nmid (\phi(n^2) + \sigma_2(n))$.

To prove the result in the case of $\omega(n) = 2$ and $\omega(n) = 3$, we first establish the following Lemma.

Lemma 2.2. If $x \leq \frac{1}{2}$, $y \leq \frac{1}{3}$ and $z \leq \frac{1}{5}$ and if $0 \leq z < y < x$, then,

$$(1 - x)(1 - y)(1 - z) + \frac{1}{(1 - x^2)(1 - y^2)(1 - z^2)} < 2.$$

Proof. Using the identity

$$\frac{1}{(1-x^2)(1-y^2)(1-z^2)} = 1 + \frac{x^2}{1-x^2} + \frac{y^2}{1-y^2} + \frac{z^2}{1-z^2} + \frac{x^2y^2}{(1-x^2)(1-y^2)} + \frac{y^2z^2}{(1-y^2)(1-z^2)} + \frac{z^2x^2}{(1-z^2)(1-x^2)} + \frac{x^2y^2z^2}{(1-x^2)(1-y^2)(1-z^2)},$$

we have,

$$(1-x)(1-y)(1-z) + \frac{1}{(1-x^2)(1-y^2)(1-z^2)}$$
$$= 2 + xy + yz + zx + \frac{x^2}{(1-x^2)(1-y^2)} + \frac{y^2}{(1-y^2)(1-z^2)} + \frac{z^2}{(1-z^2)(1-x^2)} + \frac{x^2y^2z^2}{(1-x^2)(1-y^2)(1-z^2)} - x - y - z - xyz. \quad (2.1)$$

Since $x \le \frac{1}{2}$, $y \le \frac{1}{3}$ and $z \le \frac{1}{5}$, we have,

$$\frac{1}{1-x^2} \le \frac{4}{3}, \quad \frac{1}{1-y^2} \le \frac{9}{8} \text{ and } \frac{1}{1-z^2} \le \frac{25}{24}.$$

Hence, $z + \frac{x}{(1-x^2)(1-y^2)}$, $x + \frac{y}{(1-y^2)(1-z^2)}$, $y + \frac{z}{(1-z^2)(1-x^2)}$ and $\frac{xyz}{(1-x^2)(1-y^2)(1-z^2)}$ all lie in the interval $(0,1)$. Employing this fact in (2.1), we see that

$$(1-x)(1-y)(1-z) + \frac{1}{(1-x^2)(1-y^2)(1-z^2)} < 2.$$

We now prove the result for the case $\omega(n) = 2$.

Theorem 2.3. Let $n > 1$ with $\omega(n) = 2$. Then, $n^2 \nmid (\phi(n^2) + \sigma_2(n))$.

Proof. Since $\omega(n) = 2$, we can suppose n to be of the form $n = p_1^r p_2^s$, where p_1 and p_2 are distinct primes with $p_1 < p_2$ and r and s are positive integers.

Consider,

$$
\begin{aligned}
&(p_1^2-1)(p_2^2-1)p_1^{2r}p_2^{2s}\left\{\frac{\phi(n^2)+\sigma_2(n)}{n^2}\right\}\\
&\quad= (p_1^2-1)(p_2^2-1)p_1^{2r}p_2^{2s}\left[\frac{(p_1-1)(p_2-1)}{p_1p_2}+\frac{(p_1^{2r+2}-1)(p_2^{2s+2}-1)}{p_1^{2r}p_2^{2s}(p_1^2-1)(p_2^2-1)}\right]\\
&\quad= (p_1-1)(p_2-1)(p_1^2-1)(p_2^2-1)p_1^{2r-1}p_2^{2s-1}+(p_1^{2r+2}-1)(p_2^{2s+2}-1)\\
&\quad< p_1^{2r+2}p_2^{2s+2}\left\{1+\frac{(p_1-1)(p_2-1)(p_1^2-1)(p_2^2-1)}{p_1^3p_2^3}\right\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\phi(n^2)+\sigma_2(n)}{n^2} &< \frac{(p_1-1)(p_2-1)}{p_1p_2}+\frac{p_1^2p_2^2}{(p_1^2-1)(p_2^2-1)}\\
&= (1-\frac{1}{p_1})(1-\frac{1}{p_2})+\frac{1}{\left(1-\frac{1}{p_1^2}\right)\left(1-\frac{1}{p_2^2}\right)}.
\end{aligned}
$$

Since $2\le p_1<p_2,$ we have, $\dfrac{1}{p_1}\le\dfrac{1}{2}$ and $\dfrac{1}{p_2}\le\dfrac{1}{3}$. Hence by Lemma 2.2, with $x=\dfrac{1}{p_1}$, $y=\dfrac{1}{p_2}$ and $z=0$, it follows that, $\dfrac{\phi(n^2)+\sigma_2(n)}{n^2}<2$. This completes the proof.

Theorem 2.4. Let $n>1$ with $\omega(n)=3$. Then, $n^2\nmid\phi(n^2)+\sigma_2(n)$.

Proof. Suppose $\omega(n)=3$. Then n can be written in the form $n=p_1^rp_2^sp_3^t$ where p_1,p_2 and p_3 are distinct primes with $p_1<p_2<p_3$ and r,s and t are positive integers. Now,

$$
\begin{aligned}
\frac{\phi(n^2)+\sigma_2(n)}{n^2} &= \frac{\phi(n)}{n}+\frac{\sigma_2(n)}{n^2}\\
&= \left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)\\
&\qquad+\frac{p_1^{2r+2}-1}{p_1^2-1}\cdot\frac{p_2^{2s+2}-1}{p_2^2-1}\cdot\frac{p_3^{2t+2}-1}{p_3^2-1}\times\frac{1}{p_1^{2r}\cdot p_2^{2s}\cdot p_3^{2t}}\\
&< \left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)+\frac{p_1^2p_2^2p_3^2}{(p_1^2-1)(p_2^2-1)(p_3^2-1)}
\end{aligned}
$$

$$= \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) + \frac{1}{\left(1 - \frac{1}{p_1^2}\right)\left(1 - \frac{1}{p_2^2}\right)\left(1 - \frac{1}{p_3^2}\right)}$$

< 2, by Lemma 2.2.

This proves the theorem.

Theorem 2.3 and Theorem 2.4 show that if $n^2 \mid (\phi(n^2) + \sigma_2(n))$, then, $\omega(n) \geq 4$. We conclude this note with the conjecture that $\mathbb{B} = \emptyset$ or equivalently there is no positive integer $n > 1$ such that $n^2 | (\phi(n^2) + \sigma_2(n))$.

References

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